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COMPUTER AIDED SOLUTION OF THE INVARIANCE EQUATION FOR TWO-VARIABLE GINI MEANS

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Abstract. Our aim is to solve the so-called invariance equation in the class of two-variable Gini means $\{G_{p,q} : p, q \in \mathbb{R}\}$, i.e., to find necessary and sufficient conditions on the 6 parameters a, b, c, d, p, q such that the identity

$$G_{p,q}\big(G_{a,b}(x, y), G_{c,d}(x, y)\big) = G_{p,q}(x, y) \qquad (x, y \in \mathbb{R}_+)$$

be valid. We recall that, for $p \neq q$, the Gini mean $G_{p,q}$ is defined by

$$G_{p,q}(x, y) := \left(\frac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}} \qquad (x, y \in \mathbb{R}_+).$$

The proof uses the computer algebra system Maple V Release 9 to compute a Taylor expansion up to 12th order, which enables us to describe all the cases of the equality.

1. Introduction

Let $\mathbb{R}_+$ denote the set of positive real numbers throughout this paper. A two-variable continuous function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ is called a *mean* on $\mathbb{R}_+$ if

$$\min(x, y) \leq M(x, y) \leq \max(x, y) \qquad (x, y \in \mathbb{R}_+) \tag{1}$$

holds. If both inequalities in (1) are strict whenever $x \neq y$, then M is called a *strict mean* on $\mathbb{R}_+$.

Given two means $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$ and $x, y \in \mathbb{R}_+$, the iteration sequence

$$\begin{aligned} x_1 &:= x, & y_1 &:= y, \\ x_{n+1} &:= M(x_n, y_n), & y_{n+1} &:= N(x_n, y_n) \qquad (n \in \mathbb{N}) \end{aligned}$$

is said to be the *Gauss-iteration* determined by the pair (M, N) with the initial values $(x, y) \in \mathbb{R}_+^2$. It is well-known (cf. [1], [2]) that if M and N are strict means then the sequences (x_n) and (y_n) are convergent and have equal limits $M \otimes N(x, y)$ which is a strict mean of the values x and y. The mean $M \otimes N$ defined by this procedure is called the *Gauss composition* of M and N.

A key result characterizing the Gauss composition of means is the following statement: Given two strict means $M, N : \mathbb{R}_+^2 \to \mathbb{R}_+$, their Gauss composition $K = M \otimes N$

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is the unique strict mean solution K of the functional equation

$$K\big(M(x,y),N(x,y)\big)=K(x,y)\qquad (x,y\in\mathbb{R}_+) \tag{2}$$

which is called the *invariance equation.*

The simplest example when the invariance equation holds is the well-known identity

$$\sqrt{xy}=\sqrt{\frac{x+y}{2}\cdot\frac{2xy}{x+y}}\qquad (x,y\in\mathbb{R}_+),$$

that is,

$$\mathcal{G}(x,y)=\mathcal{G}\big(\mathcal{A}(x,y),\mathcal{H}(x,y)\big)\qquad (x,y\in\mathbb{R}_+),$$

where $\mathcal{A},\mathcal{G}$, and $\mathcal{H}$ stand for the two-variable arithmetic, geometric, and harmonic means, respectively. Another less trivial invariance equation is the identity

$$\mathcal{A}\otimes\mathcal{G}(x,y)=\mathcal{A}\otimes\mathcal{G}\big(\mathcal{A}(x,y),\mathcal{G}(x,y)\big)\qquad (x,y\in\mathbb{R}_+),$$

where $\mathcal{A}\otimes\mathcal{G}$ denotes Gauss' *arithmetic-geometric mean* defined by

$$\mathcal{A}\otimes\mathcal{G}(x,y)=\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{x^2\cos^2 t+y^2\sin^2 t}}\right)^{-1}\qquad (x,y\in\mathbb{R}_+).$$

The invariance equation in more general classes of means has recently been studied extensively by many authors in various papers. The invariance of the arithmetic mean $\mathcal{A}$ with respect to two quasi-arithmetic means was first investigated by Matkowski [3] under twice continuous differentiability assumptions concerning the generating functions of the quasi-arithmetic means. These regularity assumptions were weakened step-by-step by Daróczy, Maksa, and Páles in the papers [4], [5], and finally this problem was completely solved assuming only continuity of the unknown functions involved [2]. The invariance equation involving three weighted quasi-arithmetic means was studied by Burai [6], [7] and Jarczyk–Matkowski [8], Jarczyk [9]. The final answer (where no additional regularity assumptions are required) has been obtained in [9]. In a recent paper, we have studied the invariance of the arithmetic mean with respect to two so-called generalized quasi-arithmetic mean under four times continuous differentiability assumptions [10]. The invariance of the arithmetic mean with respect to Lagrangian means was the subject of investigation of the paper [11] by Matkowski. The invariance of the arithmetic, geometric, and harmonic means with respect to the so-called Beckenbach–Gini means was studied by Matkowski in [12]. Pairs of Stolarsky means for which the geometric mean is invariant were determined by Błasińska-Lesk–Głazowska–Matkowski [13]. The invariance of the arithmetic mean with respect to further means was studied by Głazowska–Jarczyk–Matkowski [14] and Domsta–Matkowski [15].

An important class of two variable homogeneous means are the so-called Gini means (cf. Gini [16]). Given two parameters $p,q\in\mathbb{R}$, the two-variable mean $G_{p,q}:\mathbb{R}_+^2\to\mathbb{R}_+$ is defined by the following formula

$$G_{p,q}(x,y)=\begin{cases}\left(\dfrac{x^p+y^p}{x^q+y^q}\right)^{\frac{1}{p-q}} & \text{for } p\neq q,\\[2ex] \exp\left(\dfrac{x^p\ln x+y^p\ln y}{x^p+y^p}\right) & \text{for } p=q,\end{cases}$$

for $x,y\in\mathbb{R}_+$.

The class of Gini means is a generalization of the class of power means, since taking $q=0$, we immediately get the power (or Hölder) mean of exponent p.

The aim of this paper is to solve the invariance equation in the class of Gini means, i.e., to solve (2) when each of the means M, N, and K is a Gini mean. More precisely, we want to describe the set of all 6-tuples (a, b, c, d, p, q) such that the identity

$$G_{p,q}\big(G_{a,b}(x,y), G_{c,d}(x,y)\big) = G_{p,q}(x,y) \qquad (x, y \in \mathbb{R}_+) \tag{3}$$

holds. The main result of this paper is contained in the following theorem.

Theorem. *Let $a, b, c, d, p, q \in \mathbb{R}$. Then the invariance equation* (3) *is satisfied if and only if one of the following possibilities hold:*

(i) $a + b = c + d = p + q = 0$, *i.e., all the three means are equal to the geometric mean,*
(ii) $\{a, b\} = \{c, d\} = \{p, q\}$, *i.e., all the three means are equal to each other,*
(iii) $\{a, b\} = \{-c, -d\}$ *and* $p + q = 0$, *i.e.,* $G_{p,q}$ *is the geometric mean and* $G_{a,b} = G_{-c,-d}$,
(iv) there exist $u, v \in \mathbb{R}$ *such that* $\{a, b\} = \{u + v, v\}$, $\{c, d\} = \{u - v, -v\}$, *and* $\{p, q\} = \{u, 0\}$ *(in this case,* $G_{p,q}$ *is a power mean),*
(v) there exists $w \in \mathbb{R}$ *such that* $\{a, b\} = \{3w, w\}$, $c + d = 0$, *and* $\{p, q\} = \{2w, 0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{c,d}$ *is the geometric mean),*
(vi) there exists $w \in \mathbb{R}$ *such that* $a + b = 0$, $\{c, d\} = \{3w, w\}$, *and* $\{p, q\} = \{2w, 0\}$ *(in this case,* $G_{p,q}$ *is a power mean and* $G_{a,b}$ *is the geometric mean).*

As an obvious consequence, we obtain the following solution for the so-called Matkowski–Sutô equation, i.e., when $G_{p,q}$ is equal to the arithmetic mean in (3).

Corollary. *Let $a, b, c, d \in \mathbb{R}$. Then the Matkowski–Sutô equation*

$$G_{a,b}(x,y) + G_{c,d}(x,y) = x + y \qquad (x, y \in \mathbb{R}_+)$$

is satisfied if and only if one of the following possibilities hold:

(i) $\{a, b\} = \{c, d\} = \{1, 0\}$, *i.e., the two means are equal to the arithmetic mean,*
(ii) there exist $v \in \mathbb{R}$ *such that* $\{a, b\} = \{1 + v, v\}$, $\{c, d\} = \{1 - v, -v\}$,
(iii) $\{a, b\} = \{\frac{3}{2}, \frac{1}{2}\}$ *and* $c + d = 0$ *(in this case,* $G_{c,d}$ *is the geometric mean),*
(iv) $a + b = 0$ *and* $\{c, d\} = \{\frac{3}{2}, \frac{1}{2}\}$ *(in this case,* $G_{a,b}$ *is the geometric mean).*

In view of the homogeneity of Gini means, identity (3) is equivalent to the equation

$$\frac{G_{p,q}\big(G_{a,b}(\mathrm{e}^x, \mathrm{e}^{-x}), G_{c,d}(\mathrm{e}^x, \mathrm{e}^{-x})\big)}{G_{p,q}(\mathrm{e}^x, \mathrm{e}^{-x})} = 1 \qquad (x \in \mathbb{R}). \tag{4}$$

The main and simple idea of the proof of our Theorem is to compute the Taylor expansion of the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) := \frac{G_{p,q}\big(G_{a,b}(\mathrm{e}^x, \mathrm{e}^{-x}), G_{c,d}(\mathrm{e}^x, \mathrm{e}^{-x})\big)}{G_{p,q}(\mathrm{e}^x, \mathrm{e}^{-x})} \tag{5}$$

at the point $x = 0$. Using the symmetry of Gini means, it also follows that F is an even function. Therefore all coefficients C_k of odd order in the Taylor expansion of F are equal to zero automatically. The validity of (4) yields that all of the even order coefficients also vanish. To derive the necessity of the conditions in our Theorem, we will only use the equalities $C_2 = \cdots = C_{12} = 0$, which produce 6 equations for the 6 unknown parameters a, b, c, d, p, q. The sufficiency of the conditions so obtained will be proved by a simple argument which, implicitely, shows that the equalities $C_2 = \cdots = C_{12} = 0$ yield $C_{2k} = 0$ for $k \geq 7$. Unfortunately, these 6 equations are so complicated that their solution requires the power of the Maple computer algebra

package. This will be done in the next section, where we also show the sequence of Maple commands that were performed during our computation. If the interested reader executes all these commands in a Maple worksheet, then all the computations can be repeated and checked again. During this computation, we analyze each of the Taylor coefficients separately and use the information so obtained at the subsequent steps.

2. The proof of Theorem

First we recall the characterization of the equality of two variable Gini means.

Lemma. *(Cf. [17]) Let $a, b, c, d \in \mathbb{R}$. Then the identity*

$$G_{a,b}(x, y) = G_{c,d}(x, y) \qquad (x, y \in \mathbb{R}_+) \tag{6}$$

holds if and only if one of the folowing possibilities is valid:

(i) $a + b = c + d = 0$ and, in this case, the two means are equal to the geometric mean,
(ii) $\{a, b\} = \{c, d\}$.

Proof of the Theorem. Assume that (3) holds. Then the function F defined by (5) is identically 1 on $\mathbb{R}$. Thus, for the kth-order Taylor coefficient C_k defined below, we have:

$$C_k := \frac{F^{(k)}(0)}{k!} = 0 \qquad (k \in \mathbb{N}). \tag{7}$$

Since F is even, thus $C_k = 0$ for all odd $k \in \mathbb{N}$.

In view of the symmetry of the Gini means in the parameters, we may assume that $a \geq b$, $c \geq d$, and $p \geq q$ in the sequel.

For the calculations in Maple, define the Gini mean $G_{p,q}(x, y) = G(p, q, x, y)$ and the function F given by (5) by performing the following commands:

```
> G:=(p,q,x,y)->((x^p+y^p)/(x^q+y^q))^(1/(p-q));
  F:=x->(G(p,q,G(a,b,exp(x),exp(-x)),G(c,d,exp(x),exp(-x))))
     /(G(p,q,exp(x),exp(-x)));
```

This yields the following output:

$$G := (p, q, x, y) \mapsto \left(\frac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}}$$

$$F := x \mapsto \frac{G\big(p, q, G(a, b, \mathrm{e}^x, \mathrm{e}^{-x}), G(c, d, \mathrm{e}^x, \mathrm{e}^{-x})\big)}{G(p, q, \mathrm{e}^x, \mathrm{e}^{-x})}$$

In the first step, we evaluate the 2nd-order Taylor coefficient C_2 by executing

```
> C[2]:=simplify(coeftayl(F(x),x=0,2));
```

This results

$$C_2 := \frac{1}{4}a + \frac{1}{4}b + \frac{1}{4}c + \frac{1}{4}d - \frac{1}{2}p - \frac{1}{2}q$$

Therefore, by $C_2 = 0$, we obtain our first necessary condition: $(a + b + c + d)/4 = (p+q)/2$. If one tries to compute $C_4, C_6, \ldots$, then the expressions obtained are so complicated that it is hard to get further information. In order to simplify the evaluation

of the higher-order Taylor coefficients, we introduce the following notations

$$\begin{aligned} w :=& \frac{a+b+c+d}{4} = \frac{p+q}{2}, \\ v :=& \frac{a+b-(c+d)}{4}, \\ t :=& \left(\frac{p-q}{2}\right)^2, \\ r :=& \frac{(a-b)^2+(c-d)^2}{8}, \\ s :=& \frac{(a-b)^2-(c-d)^2}{8}. \end{aligned}$$

(In the definition of w we utilized the condition $C_2 = 0$.) Then we can express the parameters a, b, c, d, p, q in the following form:

```
> a:=w+v+sqrt(r+s); b:=w+v-sqrt(r+s);
  c:=w-v+sqrt(r-s); d:=w-v-sqrt(r-s);
  p:=w+sqrt(t); q:=w-sqrt(t);
```

$$a := w + v + \sqrt{r+s}$$

$$b := w + v - \sqrt{r+s}$$

$$c := w - v + \sqrt{r-s}$$

$$d := w - v - \sqrt{r-s}$$

$$p := w + \sqrt{t}$$

$$q := w - \sqrt{t}$$

Now we evaluate the 4th order Taylor coefficient by inputting:

```
> C[4]:=simplify(coeftayl(F(x),x=0,4));
```

Then we obtain

$$C_4 := \frac{1}{3}tw - \frac{1}{3}vs - \frac{1}{3}wr$$

The condition $C_4 = 0$ yields that $wt = wr + vs$.

If $w = 0$, then $p + q = 0$ and hence $G_{p,q}$ is equal to the geometric mean. Therefore, the invariance equation (3) can be rewritten as

$$G_{a,b}(x, y)G_{c,d}(x, y) = xy \qquad (x, y \in \mathbb{R}_+).$$

This results

$$G_{a,b}(x, y) = \frac{1}{G_{c,d}(1/x, 1/y)} = G_{-c,-d}(x, y) \qquad (x, y \in \mathbb{R}_+).$$

Using the Lemma again, this identity yields that either $a + b = c + d = 0$ or $\{a, b\} = \{-c, -d\}$ must hold. Together with $p + q = 0$, these equations show that either condition (i) or condition (iii) of our theorem must be satisfied. Conversely, if conditions (i) or (iii) hold then an easy computation yields that (3) is satisfied.

In the rest of the proof, we assume that w is not zero. Then, we can express t in terms of w, v, r, s:

```
> t:=r+v*s/w;
```

$$t := r + \frac{vs}{w} \tag{8}$$

Next, we evaluate the 6th order Taylor coefficient:

```
> C[6]:=simplify(coeftayl(F(x),x=0,6));
```

$$C_6 := \frac{-2(-3w^2s^2 + 3v^2s^2 - 15w^2rv^2 - 5w^3sv - 10v^3sw + 15w^4v^2)}{45w}$$

If $v = 0$, then the condition $C_6 = 0$ simplifies to $w^2s^2 = 0$, whence $s = 0$ follows. Therefore (8) yields $t = r$ and we obtain that $a = c = p$ and $b = d = q$ which means that condition (ii) of our theorem must be fulfilled. In this case, it is obvious that (3) is satisfied.

In the rest of the proof, we assume that v is also not zero. Observe that the 6th order coefficient C_6 does not involve higher-order powers of r. Therefore, the equation $C_6 = 0$ can be solved for r. Temporarily, we denote this solution by R:

```
> R:= (15*w^4*v^2-3*w^2*s^2+3*v^2*s^2-5*w^3*v*s-10*w*v^3*s)/(15*w^2*v^2);
```

$$R := \frac{15w^4v^2 - 3w^2s^2 + 3v^2s^2 - 5w^3vs - 10wv^3s}{15w^2v^2} \tag{9}$$

Finally, we evaluate the 13th order Taylor polynomial of $F(x)$ at $x = 0$ (the Maple output is suppressed by putting `:` instead of `;` to the end of the Maple command, for the sake of brevity), then we extract the 8th, 10th and 12th order Taylor coefficients, replace r by R and factorize the expressions so obtained by inputting:

```
> T:=simplify(taylor(F(x),x=0,13)):
  for i from 8 to 12 by 2
      do C[i]:=simplify(subs(r=R,simplify(coeff(T,x,i))),factor) od;
```

$$C_8 := \frac{(v-w)(v+w)s}{70875w^3v^2}\big(2100w^3v^5 - 3850w^2v^4s + 4200w^5v^3 - 255wv^3s^2 + 153v^2s^3 - 9245w^4v^2s - 7395w^3vs^2 - 153w^2s^3\big)$$

$$C_{10} := \frac{2(v-w)(v+w)s}{1063125w^5v^4}\big(28500w^5v^9 - 59675w^4v^8s + 34470w^3v^7s^2 + 20100w^7v^7 - 299575w^6v^6s - 4260w^2v^6s^3 - 73200w^5v^5s^2 - 930wv^5s^4 + 66600w^9v^5 + 4805w^4v^4s^3 - 286500w^8v^4s + 279v^4s^5 - 169020w^7v^3s^2 - 16740w^3v^3s^4 - 558w^2v^2s^5 + 45955w^6v^2s^3 + 17670w^5vs^4 + 279w^4s^5\big)$$

$$C_{12} := \frac{2(v-w)(v+w)s}{2631234375w^7v^6}\big(-3272692500w^{10}v^8s + 22181100w^3v^9s^4 - 54365475w^6v^8s^3 - 25317375w^8v^6s^3 + 335826w^4v^2s^7 - 559710wv^7s^6 - 215221875w^6v^{12}s + 22875570w^6v^4s^5 + 16977870w^5v^3s^6 - 7649370w^3v^5s^6 - 1246797750w^8v^{10}s - 34684335w^8v^2s^5 - 777170000w^{11}v^5s^2 - 159926550w^7v^5s^4 - 335826w^2v^4s^7 + 641072375w^{10}v^4s^3 + 270963000w^5v^{11}s^2 - 1046615000w^9v^7s^2 + 11659365w^4v^6s^5 - 133190250w^5v^7s^4 - 1967022000w^{12}v^6s + 177650700w^9v^3s^4 - 8768790w^7vs^6 + 385915750w^7v^9s^2 + 149400w^2v^8s^5 - 98002025w^4v^{10}s^3 + 76725000w^7v^{13} - 57172500w^9v^{11} - 478665000w^{11}v^9 + 188100000w^{13}v^7 - 111942w^6s^7 + 111942v^6s^7\big)$$

(In the entire Maple computation, this is the only step which requires a considerable processing time. On a DualCore 2.6 GHZ processor, the elapsed time was less than 1 minute.)

C_8, C_{10} and C_{12} are obviously zero if $s(v-w)(v+w) = 0$. Thus, we have to consider the three subcases: $s = 0$, $v = w$, and $v = -w$.

In the case $s = 0$, (8) and (9) imply that $t = r = w^2$. Then we get that

$$\{a, b\} = \{2w + v, v\}, \qquad \{c, d\} = \{2w - v, -v\}, \qquad \{p, q\} = \{2w, 0\},$$

i.e., condition (iv) holds with $u := 2w$. Conversely, if condition (iv) holds and $u \neq 0$, then we have

$$\begin{aligned} G_{p,q}\big(G_{a,b}(x,y), G_{c,d}(x,y)\big) &= G_{u,0}\big(G_{u+v,v}(x,y), G_{u-v,-v}(x,y)\big) \\ &= \left(\frac{x^{u+v}+y^{u+v}}{2(x^v+y^v)} + \frac{x^{u-v}+y^{u-v}}{2(x^{-v}+y^{-v})}\right)^{\frac{1}{u}} = \left(\frac{x^{u+v}+y^{u+v}}{2(x^v+y^v)} + \frac{x^u y^v+y^u x^v}{2(x^v+y^v)}\right)^{\frac{1}{u}} \\ &= \left(\frac{x^u+y^u}{2}\right)^{\frac{1}{u}} = G_{u,0}(x,y) = G_{p,q}(x,y). \end{aligned} \tag{10}$$

Thus (3) is satisfied if $u \neq 0$. If $u = 0$, then the parameters also fulfil condition (iii), hence (3) holds in this case, too.

If $v = w$, then, by (8) and (9), $r = w^2 - s$ and $t = r + s = w^2$, respectively. Hence,

$$\{a, b\} = \{3w, w\}, \qquad c + d = 0, \qquad \{p, q\} = \{2w, 0\},$$

i.e., condition (v) holds. Conversely, if condition (v) holds and $w \neq 0$, then, using the identity (10) with $u := 2w$, $v := w$, we have

$$\begin{aligned} G_{p,q}\big(G_{a,b}(x,y), G_{c,d}(x,y)\big) &= G_{2w,0}\big(G_{2w+w,w}(x,y), G_{2w-w,-w}(x,y)\big) \\ &= G_{2w,0}(x,y) = G_{p,q}(x,y), \end{aligned}$$

which shows that (3) is fulfilled. If $w = 0$, then all the three means are geometric means, and hence (3) holds trivially.

The last case when $v = -w$ holds, similarly to the case $v = w$, implies that condition (vi) is valid. If condition (vi) holds, then (3) can also be verified.

In the rest of the proof, we can assume that $s(v+w)(v-w)$ is not zero. The Taylor coefficients C_8, C_{10}, and C_{12} are of the form

$$C_8 = \frac{(v-w)(v+w)s}{70875w^3v^2}P_8, \quad C_{10} = \frac{2(v-w)(v+w)s}{1063125w^5v^4}P_{10}, \quad C_{12} = \frac{2(v-w)(v+w)s}{2631234375w^7v^6}P_{12},$$

where P_8, P_{10}, and P_{12} are polynomials of the variables v, w, s. They can be obtained by the following Maple commands (whose output is suppressed):

```
>  P[8]:=op(5,C[8]): P[10]:=op(5,C[10]): P[12]:=op(5,C[12]):
```

The equalities $C_8 = C_{10} = C_{12} = 0$ and $s(v+w)(v-w) \neq 0$ imply that $P_8 = P_{10} = P_{12} = 0$. In what follows, we show that there is no solution v, w, s to this system of equations.

The variable s is a common root of the polynomials P_8 and P_{10}. Therefore the resultant $R_{8,10}$ of these two polynomials (with respect to s) is zero:

```
> R[8,10]:=factor(resultant(P[8],P[10],s));
```

$$\begin{aligned}R_{8,10} :=&136687500w^{15}v^{15}(v-w)^2(v+w)^2\\&\big(1178440166794705680\,v^{18}-34849488132334981400\,w^2v^{16}\\&+270956577734769761 50\,w^4v^{14}+2157163953185024831539\,v^{12}w^6\\&+19335728720363587723895\,w^8v^{10}+77098340762854904758838\,w^{10}v^8\\&+135541716064734053550290\,w^{12}v^6+52974528518488497499557\,v^4w^{14}\\&+21000340485870092609 85\,w^{16}v^2+444986124077664744 66\,w^{18}\big)\end{aligned}$$

Since $vw(v-w)(v+w)\neq 0$ holds, we get that v and w are solutions of a homogeneous two variable polynomial equation of degree 18. Writing w in the form

```
> w:=z*v;
```

$$w := zv$$

we get that z is a root of a 18th degree polynomial $P_{8,10}$ (more usefully, z^2 is a root of a 9th degree polynomial), where:

```
> P[8,10]:=simplify(op(4,R[8,10])/v^18);
```

$$\begin{aligned}P_{8,10} :=&1178440166794705680-34849488132334981400z^2\\&+270956577734769761 50z^4+2157163953185024831539z^6\\&+19335728720363587723895z^8+77098340762854904758838z^{10}\\&+135541716064734053550290z^{12}+52974528518488497499557z^{14}\\&+21000340485870092609 85z^{16}+444986124077664744 66z^{18}.\end{aligned}$$

The variable s is also a common root of the polynomials P_8 and P_{12}. Therefore the resultant $R_{8,12}$ of these two polynomials (with respect to s) is again zero:

```
> R[8,12]:=factor(resultant(P[8],P[12],s)):
```

Since $vw(v-w)(v+w)\neq 0$ holds, we now get that v and w are solutions of a homogeneous two variable polynomial of degree 26, whence we get that z is a root of the 26th degree polynomial $P_{8,12}$, where:

```
> P[8,12]:=simplify(op(4,R[8,12])/v^26);
```

$$\begin{aligned}P[8,12] :=& 819606370059538387170109 1232-179090512353635410423157248720z^2\\&-2262574745604112043731392907114z^4+11198535065282946302316347517923z^6\\&+36907535586106509075339608582 4722z^8+33212032129660632198000142045396 94z^{10}\\&+170182211685973585913283463586401 28z^{12}+55161742271395394206883716537690208z^{14}\\&+113024609788553283598449985201081964z^{16}+136472191224999845881431378284988722z^{18}\\&+83840233563357841801204648333566258z^{20}+19391722782753178903737004919064981z^{22}\\&+1234978033803167388960240130106010z^{24}+957110507396052105484004422039 92z^{26}\end{aligned}$$

Now computing the resultant of the two polynomials $P_{8,10}$ and $P_{8,12}$ by

```
> Q:=resultant(P[8,10],P[8,12],z):
```

it follows that Q is a (huge) nonzero number, hence $P_{8,10}$ and $P_{8,12}$ cannot have a common root. This proves that $R_{8,10}$ and $R_{8,12}$ can be simultaneously zero if and only if $vw(v-w)(v+w)=0$ holds. Hence $C_8=C_{10}=C_{12}=0$ can only hold also in this case. $\square$

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